ChE 515: Problems on Tensor Alegbra

1. Let $\mathbf{a}$ and $\mathbf{b}$ be given three-dimensional vectors and $\mathbf{x}$ unknown. Without introducing components, show that the unique solution of the linear algebraic equations

$$\mathbf{x} + \mathbf{a} \times \mathbf{x} = \mathbf{b} \tag{1}$$

is

$$\mathbf{x} = \frac{\mathbf{b} + (\mathbf{a}.\mathbf{b})\mathbf{a} - \mathbf{a} \times \mathbf{b}}{1 + \mathbf{a}.\mathbf{a}}. \tag{2}$$

Hint: Let $\mathbf{x} = A\mathbf{a} + B\mathbf{b} + C\mathbf{a} \times \mathbf{b}$. Explain why a solution of this form can be sought.

2. Without introducing components, show that $(\mathbf{uv})^T = \mathbf{vu}$.

3. Express the velocity gradient tensor $\nabla \mathbf{u}$ as the sum of symmetric and skew-symmetric tensors.

4. Express the the cross product operator, $\mathbf{u}\times$, as a second order tensor. Find its Cartesian components.

5. Let ρ and $\mathbf{v}$ denote, respectively, the density and velocity of a fluid at a given point P at a given time t. Π is a plane with unit normal $\hat{\mathbf{n}}$ passing through P. The momentum flux across the plane Π at point P and time t is defined as $\rho\mathbf{vv}.\hat{\mathbf{n}}$, i.e., $\rho\mathbf{vv}.\hat{\mathbf{n}}dA\,dt$ is the momentum at P and t carried across a differential element of area dA oriented along $\hat{\mathbf{n}}$ in time dt. $\rho\mathbf{vv}$ is called the *momentum flux tensor* at P and t.

(a). If $\mathbf{v} \sim (v_x, v_y, v_z)$, find the Cartesian components of $\rho\mathbf{vv}$.

(b) If $\mathbf{v} \sim (3, -1, 2)$ and $\rho = 4$ at a given point and time, determine the momentum flux across the plane with normal $\mathbf{n} \sim (-1, 1, 3)$.

6. A second order tensor $\mathbf{T}$ is said to be *positive definite* if

$$\mathbf{x}.\mathbf{Tx} > 0 \quad \forall \mathbf{x} \neq \mathbf{0}$$

(a). Show that if $\mathbf{T}$ is positive definite, then all the eigenvalues of $\mathbf{T}$ must be positive.

(b). Show that the momentum flux tensor (problem 5) is positive definite.

7(a). Show that the vectors $\mathbf{g}_1 \sim (1, -1, 2)$, $\mathbf{g}_2 \sim (0, 1, 1)$ and $\mathbf{g}_3 \sim (-1, -2, 1)$ form a basis to 3-dimensional space. Find the reciprocal basis vectors for this basis. Find the roof (contravariant) and the cellar (covariant) components of the vector $\mathbf{v} \sim (3, 3, 6)$ with respect to this basis and the reciprocal basis respectively.

(b). Let $\mathbf{u} = 2\mathbf{g}_1 - \mathbf{g}_2 + 4\mathbf{g}_3$ and $\mathbf{v} = 2\mathbf{g}_1 + 3\mathbf{g}_2 - \mathbf{g}_3$ for $\mathbf{g}_i$ given in part (a). Using the definition of the cross product, $\mathbf{u} \times \mathbf{v} = \epsilon_{ijk} u^i v^j \mathbf{g}^k$, compute the cellar components of $\mathbf{u} \times \mathbf{v}$. Note that the non-zero components of the permutation tensor is not ± 1 but $\pm J$ where J is the Jacobian associated with the basis.

8. Given a basis $\mathbf{g}^i$, the third order permutation tensor may be denoted and defined by $P \equiv \epsilon_{ijk}\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k$. Show that
(a). $\mathbf{P}...\mathbf{wvu} = \mathbf{u}.(\mathbf{v} \times \mathbf{w})$
(b). $\mathbf{P}..\mathbf{vu} = \mathbf{u} \times \mathbf{v}$
(c). $\mathbf{P}.\mathbf{u} = -\mathbf{u}\times$

9. An orthogonal tensor $\mathbf{Q}$ satisfies the relationship $\mathbf{Q}^t\mathbf{Q} = \mathbf{1}$. In other words, the inverse of $\mathbf{Q}$ is the same as its transpose.
(a). Let $Q = [\mathbf{e}_i.\mathbf{Q}\mathbf{e}_j]$, where $\mathbf{e}_i$ is the standard Cartesian basis. Show that $Q^tQ = I$.
(b). Show that $\det(Q) = \pm 1$
(c). Show that an orthogonal transoformation of the basis does not change the components of the identity tensor.
(d). An example of the orthogonal tensor is the reflection tensor $\mathbf{H}$ that reflects a vector across a plane with unit normal $\mathbf{n}$. Note that $\mathbf{H} = \mathbf{1} - \mathbf{2nn}$. Verify (i). $\mathbf{H}^2 = \mathbf{1}$ and (ii). $\det(\mathbf{H}) = 1$.

Additional Problems (Solutions to these need not be submitted)

10. Analogous to the reflection tensor is the rotator $\mathbf{R}$ that is characterized by an axis of rotation with direction $\mathbf{e}$ and an angle of rotation θ (taken as positive or negative based on the right-hand rule). Consider the action of $\mathbf{R}$ on vector $\mathbf{u}$ that rotates it to give vector $\mathbf{v}$.
(a). By resolving $\mathbf{v}$ into three mutually perpendicular directions given by $\mathbf{e}$, $\mathbf{e} \times \mathbf{u}$ and $\mathbf{e} \times \mathbf{uu}$, show that $\mathbf{R} = \cos(\theta)\mathbf{1} + (\mathbf{1} - \cos(\theta))\mathbf{ee} + \sin(\theta)\mathbf{e}\times$
(b). Show that $\det(\mathbf{R}) = \pm 1$. Find $\mathbf{R}^t$.

11. The product $\epsilon^{ijk}\epsilon_{pqr}$ can be expressed as the determinant of $A = C1, C2, C3$ where $C1 = (\delta^i_p, \delta^j_p, \delta^k_p)^t$, $C2 = (\delta^i_q, \delta^j_q, \delta^k_q)^t$ and $C3 = (\delta^i_r, \delta^j_r, \delta^k_r)^t$. Show that $\epsilon^{ijk}\epsilon_{pqk} = \delta^i_p\delta^j_q - \delta^i_q\delta^j_p$.

12. Generalize the above result for $\epsilon^{ijk}\epsilon_{pqr}\det(A)$ where $A = [A^i_j]$ is a 3×3 matrix. Show that (the Cayley-Hamilton theorem) $A^3 - (traceA)A^2 + (1/2)[(traceA)^2 - traceA^2]A - det(A)I = 0$.